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On derivation of spectral sensitivities of the human cones from trichromatic colour matching functions

Alexander D. Logvinenko *

School of Psychology, Queen's University Belfast, Belfast BT9 5BP, UK

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Abstract

Despite the recent advance made by using the direct methods of retinal densitometry, microspectrophotometry and suction electrophysiology, the psychophysical approach based on colour matching data still remains an important source of accurate information about the spectral sensitivity of the cone photoreceptors in the human visual system. However, the commonly used technique of estimating cone sensitivities, based on the assumption that dichromacy is caused by the lack of one of the three types of the cone photoreceptors, requires the colour matching functions not only from trichromatic observers but from dichromats as well. Here we evaluate an alternative approach, originally put forward by Bongard and Smirnov, that derives cone spectral sensitivities from colour matching functions only; without resorting to colour deficiency or any other data. When applied to CIE standard colour matching functions, this method yields curves of spectral sensitivities that are close to the classical Smith–Pokorny fundamentals, though the long-wave cone is shifted towards the short-wave region of the spectrum by 5 nm, as compared with Smith and Pokorny's results. © 1998 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In classical trichromatic colour theory [1], metameric colour matching is assumed to be based on three linear colour mechanisms, φ_1, φ_2 and φ_3, which are generally believed to represent the spectral sensitivities of the human cone photoreceptors (the cone fundamentals). More specifically, any two coloured lights a and b match if $\varphi_1(a)=\varphi_1(b)$, $\varphi_2(a)=\varphi_2(b)$ and $\varphi_3(a)=\varphi_3(b)$. It follows that there exist three such lights e_1, e_2 and e_3 (primaries) that for any other light x

$$\begin{aligned}\varphi_1(x) &= \mathrm{CMF}_1(x)\varphi_1(e_1)+\mathrm{CMF}_2(x)\varphi_1(e_2)\\&\quad+\mathrm{CMF}_3(x)\varphi_1(e_3),\\ \varphi_2(x) &= \mathrm{CMF}_1(x)\varphi_2(e_1)+\mathrm{CMF}_2(x)\varphi_2(e_2)\\&\quad+\mathrm{CMF}_3(x)\varphi_2(e_3),\\ \varphi_3(x) &= \mathrm{CMF}_1(x)\varphi_3(e_1)+\mathrm{CMF}_2(x)\varphi_3(e_2)\\&\quad+\mathrm{CMF}_3(x)\varphi_3(e_3). \end{aligned} \tag{1}$$

Or in matrix form

$$\begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\\ \varphi_3(x)\end{pmatrix} = \begin{pmatrix}\varphi_1(e_1) & \varphi_1(e_2) & \varphi_1(e_3)\\ \varphi_2(e_1) & \varphi_2(e_2) & \varphi_2(e_3)\\ \varphi_3(e_1) & \varphi_3(e_2) & \varphi_3(e_3)\end{pmatrix}\begin{pmatrix}\mathrm{CMF}_1(x)\\ \mathrm{CMF}_2(x)\\ \mathrm{CMF}_3(x)\end{pmatrix}. \tag{1a}$$

Here CMF_i stands for the ith colour matching function which is determined from colour matching experiments. As we can see, the three cone fundamentals, underlying colour matching for trichromats, are based on the linear transformation of the colour matching functions. However, the matrix of this transformation, i.e. the response matrix, is unknown, unless the cone fundamentals are specified beforehand.

In one particular case, when the primaries constitute what is called the dual (or cardinal) basis (see more about the concept of the dual basis in vision in Refs. [2,3]), i.e. when

* Fax: +44 232 664144; e-mail: a.logvinenko@qub.ac.uk.

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$$\varphi_i(e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (2)$$

the colour matching functions coincide with the cone fundamentals. The cone fundamentals uniquely determine the colour co-ordinates of the dual basis in the colour vector space; and vice versa. Given the colour co-ordinates of the dual basis, one can calculate the cone fundamentals from the colour matching functions.

If the cardinal directions rather than the cardinal vectors themselves are known, then the response matrix $\{\varphi_i(e_j)\}$ has a diagonal form with unknown numbers on the diagonal. In this case, the colour matching functions can be recovered up to scaling factors. The cardinal directions can be specified by the points on a chromaticity diagram where the cardinal directions intersect the diagram plane.

It is believed that the chromaticity co-ordinates of these points (let us call them cardinal) can be found by using colour deficiency data, such as dichromatic colour matches. Dichromatic observers are individuals with hereditary colour defects who can perform colour matching for the full spectrum with only two primaries. It is generally assumed that dichromats lack one type of cone photoreceptors—longwave (protanope), middlewave (deuteranope), or shortwave (tritanope). Koenig was probably the first to suggest that given the colour matching functions from dichromats of all three types, one can determine cardinal points corresponding to the cone fundamentals missing in these dichromats [1]. In this case they are called confusion (or copunctal) points. Then, the confusion points averaged over a population of dichromats of the same type are used to transform the averaged colour matching functions from normal trichromats, e.g. the CIE colour matching functions.

However, cone fundamentals derived by using such an approach, though widely accepted, can be regarded only as a first rough approximation. The problem is that while the confusion points specify the cardinal directions of those particular colour blind individuals from whom they were derived, they are then used to derive the cone fundamentals for other individuals (with normal colour vision). Strictly speaking, for the cone fundamentals to be derived from colour matching functions measured for some individual, one needs to know the cardinal points for this particular individual rather than for observers who are colour-blind. Besides that, the validity of the method of deriving confusion points rests heavily upon the main assumption of Koenig's theory of colour blindness, namely, that dichromats are a reduced form of trichromats.

On the other hand, as far back as 1954, Bongard and Smirnov presented an elegant technique which permits one to derive cone fundamentals from individual colour matching functions without resorting to either a theory of colour blindness or colour deficiency data. The rationale of this alternative approach was as follows. Note, first, that for a cone fundamental to coincide with a colour matching function, the requirement that the set of primaries has to diagonalise the response matrix $\{\varphi_i(e_j)\}$, is excessively strict. A particular cone fundamental will coincide with one of the colour matching functions up to a scaling factor if the primaries e_1, e_2, e_3 are such, that this cone fundamental has a non-zero response to only one of them. Indeed, let us consider a cone fundamental, say φ_1, and assume that $\varphi_1(e_1) \neq 0$ and $\varphi_1(e_2) = \varphi_1(e_3) = 0$. In this case, it follows from the first equation in Eq. (1) that $\mathrm{CMF}_1(x) = \varphi_1(x)/\varphi_1(e_1)$. As pointed out by Bongard and Smirnov [4], this fact can be used to derive the cone fundamentals provided that they do not overlap across the spectrum.

Some evidence in favour of the non-overlapping assumption can be found in the CIE chromaticity diagram (Fig. 1(a)). As can be seen, all spectral stimuli from the long-wavelength (above 600 nm) end of the visible spectrum fall along a straight line. This implies that monochromatic lights from this end region activate only two cone fundamentals, and that the spectral sensitivity curves for the three cone fundamentals fail to overlap at this region of the spectrum. Undoubtedly, it is the short-wavelength (S) cones that monochromatic lights from the long-wavelength range (λ_2, $\lambda_3 > 600$ nm) do not activate. Therefore, the S cone fundamental can be derived from the colour matching functions based on a set of monochromatic primaries, two of which are taken from the long-wavelength range of spectrum.

Although the assumption of non-overlapping does not hold true for the short-wavelength end of the visible part of spectrum, the approach can still be applied to evaluate the long-wavelength (L) and medium-wavelength (M) fundamentals since it remains effective even when the response of the cone fundamental to be determined for two of the primaries is only approximately equal to zero. As can be seen in Eq. (1), if $\varphi_1(e_2) \approx 0$ and $\varphi_1(e_3) \approx 0$ then $\mathrm{CMF}_1(x) \approx \varphi_1(x)/\varphi_1(e_1)$; and the bigger the ratios $\varphi_1(e_1)/\varphi_1(e_2)$ and $\varphi_1(e_1)/\varphi_1(e_3)$, the more accurate the evaluation of φ_1.

Bongard and Smirnov [4] used their method to derive the spectral sensitivities of the cones directly from the CIE standard colour matching functions. The derived cone fundamentals were claimed to be in a good agreement with those from dichromatic observers available for Bongard and Smirnov at the time [5]. Then Speranskaya and Lobanova applied the Bongard and Smirnov technique to individual colour matching functions obtained by them from normal [6] and anomalous trichromats [7]. While having some problems with estimating the L cone fundamental, they concluded that the method worked well, especially when applied to individual data, and the results were generally consistent with those derived from dichromats. Since then, this method has never been employed (at least to the au-

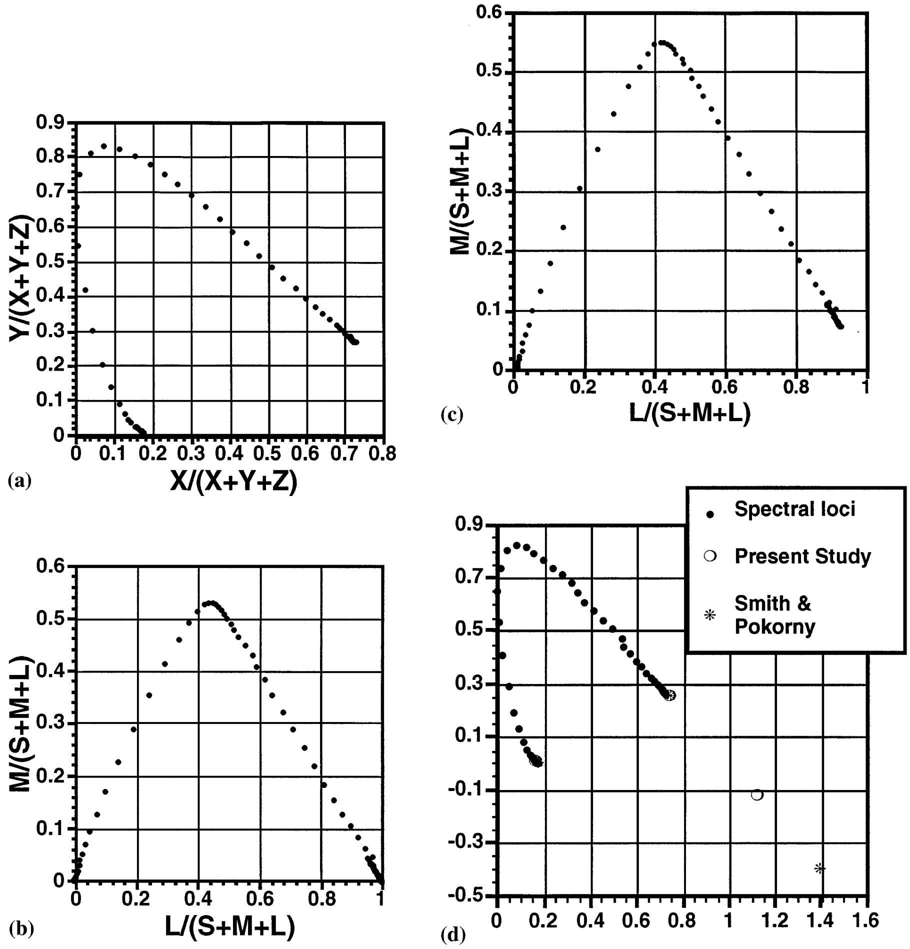

Fig. 1. (a) CIE chromaticity diagram; (b) chromaticity diagram based on the fundamentals of the present study (see Eq. (3)); (c) chromaticity diagram based on the amended fundamentals (see explanation in text); (d) CIE chromaticity diagram with Smith & Pokorny's confusion points, and the cardinal points as derived in the present study.

thor's knowledge). However, its potentials do not seem to be completely exhausted. Particularly, it would be interesting to see how the method will work when applied to the individual data obtained by Stiles and Burch in 1955 and published later [8]. However, before that, it seems worthwile—and it is the main purpose of the present report—to check the validity of the Bongard and Smirnov technique by applying it to the CIE colour matching functions, $\bar{x}(\lambda)$, $\bar{y}(\lambda)$ and $\bar{z}(\lambda)$, amended by Vos (Ref. [1]; Table I (5.5.2)) in order to compare the results with the well-known cone fundamentals derived on the basis of colour deficient vision [9,10].

Monochromatic primaries with wavelengths 380, 385 and 560 nm and 450, 550 and 690 nm, have been taken to evaluate the L and S cone fundamentals, respectively. The M cone fundamental was derived when the extremes of the visible spectrum were included into a set of primaries (namely 380, 550 and 690 nm)[1].

[1] Although we also tried many other combinations of primaries, presented here are those that yield the best result. It should be noted, however, that the values of the CIE colour matching functions for wavelengths less than 410 nm are based on extrapolation. Although it seems acceptable for the demonstration of the technique, it is clear that further analysis of this sort will require colour matching data from a broader range.

The colour matching functions, $\mathrm{CMF}_1(\lambda)$, $\mathrm{CMF}_2(\lambda)$ and $\mathrm{CMF}_3(\lambda)$, based on a particular set of monochromatic primaries with the wavelengths λ_1, λ_2, λ_3, were recalculated according to the known formula (e.g. Ref. [1], p. 129)

$$\begin{pmatrix} \mathrm{CMF}_1(\lambda) \\ \mathrm{CMF}_2(\lambda) \\ \mathrm{CMF}_3(\lambda) \end{pmatrix} = A \begin{pmatrix} \bar{x}(\lambda) \\ \bar{y}(\lambda) \\ \bar{z}(\lambda) \end{pmatrix},$$

$$\text{where } A^{-1} = \begin{pmatrix} \bar{x}(\lambda_1) & \bar{x}(\lambda_2) & \bar{x}(\lambda_3) \\ \bar{y}(\lambda_1) & \bar{y}(\lambda_2) & \bar{y}(\lambda_3) \\ \bar{z}(\lambda_1) & \bar{z}(\lambda_2) & \bar{z}(\lambda_3) \end{pmatrix}.$$

Table 1
The cardinal and confusion points

	x_{pc}	y_{pc}	x_{dc}	y_{dc}	x_{tc}	y_{tc}
Present study	0.7299	0.2701	1.1217	−0.1207	0.1776	0.0132
Present study (amended)	0.7444	0.2557	1.1220	−0.1209	0.1658	0.0071
Smith and Pokorny	0.7465	0.2535	1.4000	−0.4000	0.1748	0.0

The derived cone fundamentals are expressed in terms of the colour matching functions, $\bar{x}(\lambda)$, $\bar{y}(\lambda)$ and $\bar{z}(\lambda)$, as

$$L(\lambda) = 0.10\bar{x} + 0.94\bar{y} - 0.038\bar{z},$$
$$M(\lambda) = -0.44\bar{x} + 1.20\bar{y} + 0.078\bar{z}, \quad (3)$$
$$S(\lambda) = 0.00046\bar{x} - 0.0013\bar{y} + 0.68\bar{z}.$$

The chromaticity diagram based on the fundamentals (Eq. (3)) is depicted in Fig. 1(b). Note that (i) the chromaticities of the short- and long-wavelength lights produce more pronouncedly linear plots than the classical CIE diagram (Fig. 1(a)); (ii) the very ends of the spectrum map onto the points with co-ordinates (0, 0) and (0, 1). The latter particularly implies a negligible contribution of the L and M cones in the first case, and the S and M cones in the second. All this corroborates the main idea on which the derivation is based, and shows that the method is self-consistent.

It should be noted, however, that although the transformation (Eq. (3)) yields all three functions positive and very close to the cone fundamentals derived from colour blindness [9,10], the long-wavelength (> 600 nm) part of the M fundamental in Eq. (3) lacks a completely realistic form. But after a minor adjustment of the second equation in Eq. (3) (the weighting coefficient $\bar{y}$ has been increased from 1.20 to 1.29) it looks biologically more plausible as can be seen in Fig. 3, which shows the S, M (an amended version) and L fundamentals plotted on a logarithmic ordinate and normalised to unit sensitivity. Such an amendment does not substantially change the cardinal points corresponding to the fundamentals (see Table 1); however, after this amendment the chromaticity diagram loses its perfect triangular form (Fig. 1(c)).

As can be seen in Table 1 and Fig. 1(d), two of three cardinal points derived in the present study practically coincide with the corresponding confusion points of Smith and Pokorny. However, the cardinal point corresponding to the M cone fundamental differs considerably from the deuteranopic confusion points (x_{dc}, y_{dc}) of Smith and Pokorny. While lying on the same line in the chromaticity diagram plane, the M cardinal point is shifted towards the protanopic confusion point (x_{pc}, y_{pc}).

However, it should be noted that the cardinal points proximity to the confusion points is hardly a relevant basis for assessment of similarity between the two sets of the cone fundamentals. As a matter of fact, Euclidean distance which underlies our intuitive understanding of proximity, is not invariant under a linear transformation. So when changing the colour co-ordinates, the proximity relationship between cardinal and confusion points may change as well. To illustrate that, let us plot the derived cardinal points on the chromaticity diagram based on the Smith and Pokorny fundamentals (Fig. 2). Presented this way, the clusters of the cardinal points and confusion points look quite different. For example, the position of the M cardinal point is such that its direction in the colour diagram might even be mistaken for the red–green opponent direction. However, since metrical dimensions, such as distance and angle, depend on the frame of reference in the

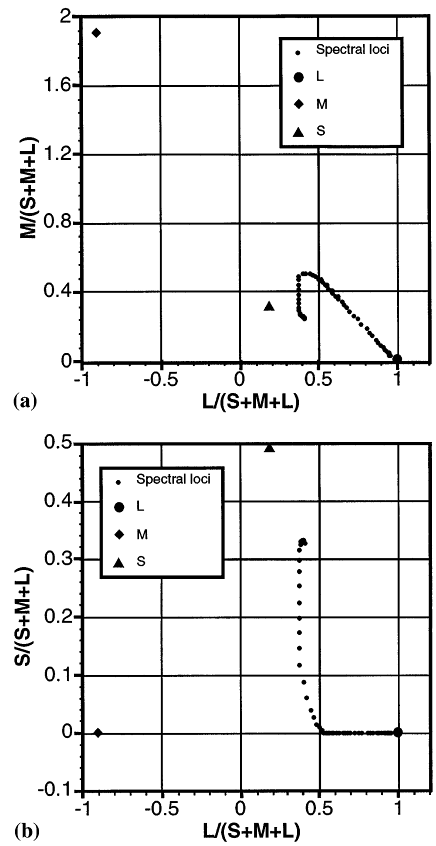

Fig. 2. Chromaticity diagrams based on the cone fundamentals of Smith and Pokorny with the cardinal points derived in the present study. The S, M, and L confusion points have the following coordinates: S (0, 0) in (a) and (1, 0) in (b); M, (1, 0) in (a) and (0, 0) in (b); L, (0, 1) in both (a) and (b).

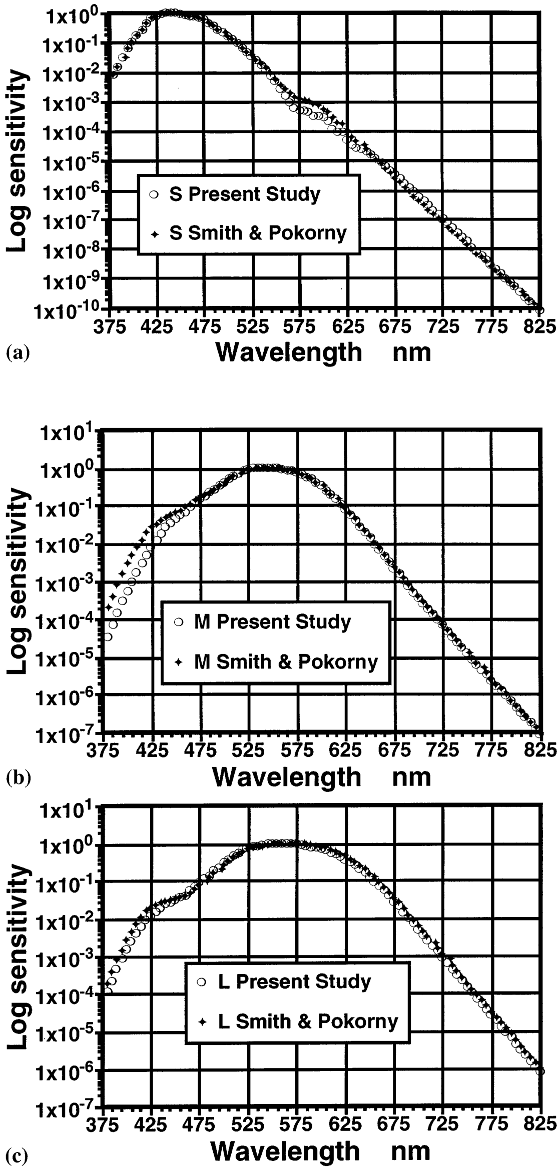

Fig. 3. Estimates of the human cone spectral sensitivities by the present study; and comparisons of the cone fundamentals of Smith and Pokorny and the present study.

colour space, there is no point in relying on an angular measure of proximity between two cardinal directions.

So, since there is no natural metric in the colour space, we should rather base our judgements on similarity of the geometrical form of curves than on relative positions of corresponding cardinal and confusion points on the colour diagram. As a matter of fact, there is quite good agreement between our results and the cone fundamentals of Smith and Pokorny obtained from dichromatic observers for S (Fig. 3(a)), and M except for the short-wavelength portion of the spectrum (Fig. 3(b)) but not for the L cone fundamental which is found to be closer to M with maximum sensitivity at 560 nm (Fig. 3(c)). However, the differences between the cone fundamentals obtained by us and by Smith and Pokorny do not seem too large, especially if one takes into account that the colour matching functions and confusion points are obtained from different populations.

In conclusion, note that the method based on dichromat colour matching provides us with averaged cone fundamentals. In addition with large variability in measuring the confusion points, it makes this method unsuitable for deriving individual, rather than averaged, cone fundamentals, not to mention evaluating individual differences in cone fundamentals. In contrast, the method outlined here can be used to derive colour fundamentals from individual colour matching functions. Furthermore, it may prove an alternative method for assessing individual difference in cone fundamentals.

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